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NON-LINEAR VIBRATIONS OF VISCOELASTIC MOVING BELTS, PART I: FREE VIBRATION ANALYSIS

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The non-linear free vibration of viscoelastic moving belts is studied. Based on the linear viscoelastic differential constitutive law, the generalized equation of motion is derived for a moving belt with geometric non-linearities. The method of multiple scales is applied directly to the governing equation which is in the form of continuous autonomous gyroscopic systems with weak non-linearity. This direct treatment does not involve a prior assumption regarding the spatial solutions. The non-linear natural frequencies and free response amplitude for autonomous systems are predicted by the perturbation method. The results obtained with the quasi-static assumption and those without this assumption are compared. The effects of elastic and viscoelastic parameters, axial moving speed, and the geometric non-linearity on natural frequencies and amplitudes of the free response are investigated from numerical examples.

1. INTRODUCTION

Moving belts used in power transmissions are an example of a class of mechanical systems commonly referred to as axially moving strings. The vibration analysis of such a system has been studied extensively. For linear vibration analysis, Skutch [1] first determined the natural frequencies of a moving string by superposition of two waves propagating in opposite directions. The classical modal analysis, which is applied to the linear non-translating string model, is not directly applicable to linear axially moving strings since the generalized co-ordinates in an eigenfunction expansion remain coupled. Wickert and Mote [2] modified the classical modal analysis method by casting the equations of motion for a travelling string into a canonical, first order form that is defined by one symmetric and one skew-symmetric matrix differential operators. When the equations of motion are represented in this form, the eigenfunctions are orthogonal with respect to each other. The response of axially moving materials to arbitrary excitation and initial conditions can be represented in closed forms.

The earliest calculation of the fundamental period of autonomous non-linear transverse vibrations of an axially moving, elastic, tensioned string was given by Mote [3]. Computation difficulties in the integration of the equation restricted the solution to speeds below 40% of the critical speed. In the work done by Thurman and Mote [4], a hybrid discretization and perturbation method were employed to quantify the speed dependence of the deviation between the linear and non-linear fundamental periods for a broad range of amplitude and speed parameters. This method was limited, however, in that secular excitation terms in the perturbation analysis were rendered small, but not eliminated.

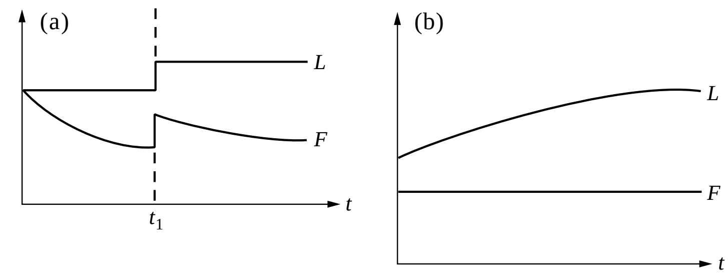

Figure 1. Principal changes in belt force and belt length with time of service for V-belts: (a) pre-tensioning with constant elongation; (b) pre-tensioning with constant force.

Bapat and Srinivasan [5] used the method of harmonic balance to obtain approximate results. In Wickert's study [6], the governing equations of motion were cast in the standard form of continuous gyroscopic systems. A second order perturbation solution was derived through the asymptotic methods of Krylov, Bogoliubov, and Mitropolsky for the near-modal response of a general gyroscopic system with weakly non-linear stiffness.

In the investigations above, the belt material is assumed to be linear elastic and damping is either ignored or introduced simply as linear viscous without reference to any damping mechanism. However, belts are usually composed of some metallic or ceramic reinforcement materials like steel-cord or glass-cord and polymeric materials such as rubber. Most of these materials exert inherently viscoelastic behavior, i.e., they flow when subjected to stress or strain. Such flow is accompanied by the dissipation of energy due to some internal loss mechanism (for example, bond breakage and bond formation reaction, dislocation). Figure 1 illustrates the creep of a practical belt during the operation [7]. Dynamic loading in operation will not only lead to creep, but also to orientation of

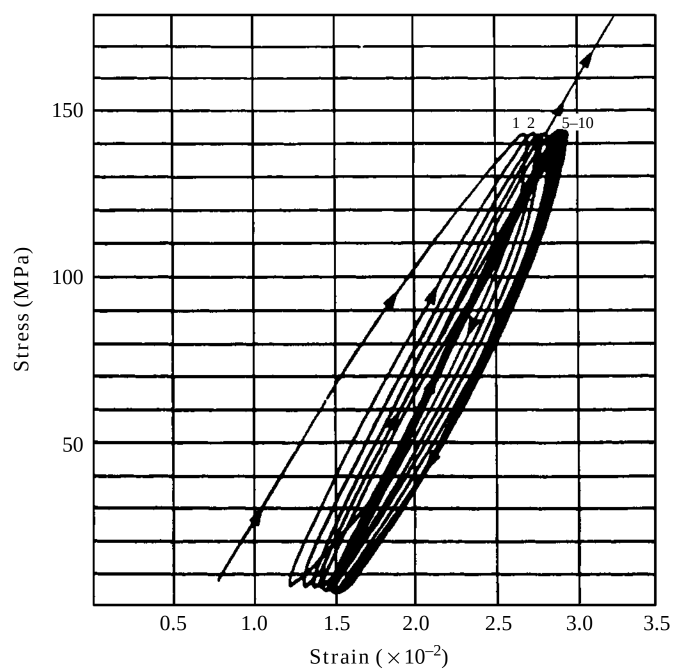

Figure 2. Stress–strain curves showing relaxation and creep effects by repeated deformation of treated polyester cord $1100 \times 2 \times 5$.

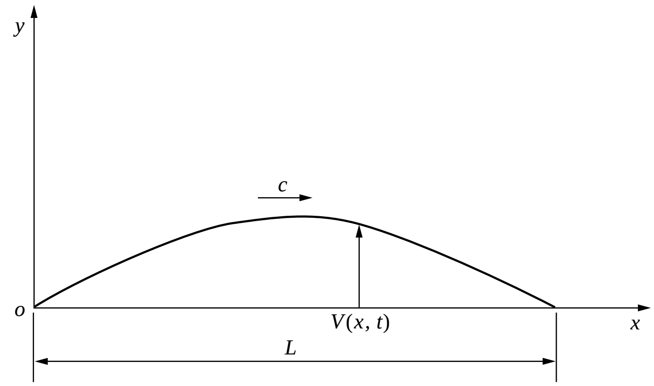

Figure 3. A prototypical model of a viscoelastic moving belt.

the material, by which its stiffnes increases. Figure 2 shows relaxation and creep effect by repeated deformations of treated polyester cord $1100 \times 2 \times 5$ [7]. The viscoelastic characteristic generally leads to reduced noise and vibrations in the accessory systems. On the other hand, it can also cause excessive slip of belts which can lead to a temperature increase. In order to accurately model the mechanical characteristics of belt materials such as creep and damping, it is necessary to turn to the viscoelastic theory of materials.

The literature that is specially related to a viscoelastic moving continuum is very limited. However, various methods have been presented for the vibration analysis of structures composed of viscoelastic materials. Findley *et al*. [8] used the correspondence and superposition principles to solve the governing equations of the viscoelastic beams. Christensen [9] employed Fourier transform to solve the transient response of viscoelastic beams. Chen [10] adopted Laplace transform and the resulting equation was solved by the finite element method.

There is only one paper by Fung [11] so far discussing the dynamic response of a viscoelastic moving string. In the paper the string material was assumed to be constituted by the hereditary integral type. The governing equation was reduced to a set of second order non-linear differential-integral equations that were then solved by the finite difference method.

In this paper, based on the linear viscoelastic differential constitutive law, the equation of motion, which is in the form of gyroscopic system, is obtained for a viscoelastic moving belt with geometric non-linearities. As a first step to tackle the problem, free vibration and forced vibration analysis is performed. A modal perturbation solution is developed in the context of the asymptotic method of multiple scales for a general continuous autonomous gyroscopic system with geometric non-linearity. The near-modal non-linear response for autonomous systems is predicted by the perturbation method. The results obtained with the quasi-static assumption and those without this assumption are compared. The effects of elastic and viscoelastic parameters, axial moving speed and the non-linear term on the response are also investigated.

2. EQUATIONS OF MOTION

A prototypical model of a viscoelastic moving belt is shown in Figure 3, where c is the transport speed of the belt, L is the length of the belt span, V is the displacement in the transverse direction. Several simplifying assumptions are made in modelling moving belts, as follows: (1) only transverse vibration in the y direction is taken into consideration; (2)

belt bending stiffness is negligible; (3) transport speed of belts, c, is constant and uniform; (4) Lagrangian strain for belt extension is employed as a finite measure of the strain; (5) the viscoelastic string is in a state of uniform initial stress.

Based on the above assumptions, the equation of motion in the y direction can be obtained by Newton's second law [11]:

$$\left(\frac{T}{A}+\sigma\right)V_{xx}+V_x\sigma_x=\rho V_{tt}, \tag{1}$$

where the subscript notation x and t denote partial differentiation with respect to spatial Cartesian co-ordinate x and time t, σ is the perturbed stress, A is the area of cross-section of the belt, ρ is the mass per unit volume, and T is the initial force.

For free vibration analysis, the system is subjected to the homogeneous boundary conditions

$$V=0 \quad \text{at } x=0 \text{ and } x=L. \tag{2}$$

For moving belts, the transverse acceleration is given by [11]

$$\frac{\mathrm{d}^2V}{\mathrm{d}t^2}=\frac{\partial^2V}{\partial t^2}+2c\,\frac{\partial^2V}{\partial x\,\partial t}+c^2\frac{\partial^2V}{\partial x^2}. \tag{3}$$

Note that in equation (3), the first term on the right side represents the local acceleration component, the second term represents the Coriolis acceleration component, and the last term represents the centripetal acceleration component.

The one-dimensional linear differential viscoelastic constitutive law can be written as

$$P\sigma(t)=Q\varepsilon(t), \tag{4}$$

where P and Q are linear differential operators with respect to the time t. In a general form these operators are expressed as

$$P=\sum_{i=0}^{p}a_i\frac{\partial^i}{\partial t^i}, \qquad Q=\sum_{i=0}^{q}b_i\frac{\partial^i}{\partial t^i}, \tag{5, 6}$$

where a_i and b_i are material constants. The number of terms used in equations (5) and (6) will depend on the viscoelastic characteristic of particular materials. The viscoelastic relationship may also be written in symbolic form

$$\sigma(t)=E^*\varepsilon(t), \tag{7}$$

where E^* is defined as

$$E^*=P/Q. \tag{8}$$

Equation (7) has to be interpreted simply as an alternative notation. As the linear differential operator E^* may be handled formally as an algebraic quantity, this notation simplifies the formulations of the problem.

In this paper, only geometric non-linearity due to finite stretching is considered. For moving belts with large amplitude, the perturbed Lagrangian strain component in the x direction related to the displacement is given by

$$\varepsilon(x,t)=\tfrac{1}{2}V_x^2. \tag{9}$$

Applying the linear differential viscoelastic constitutive law, equation (7), the perturbed stress is in the form

$$\sigma = E^*(\tfrac{1}{2}V_x^2). \tag{10}$$

Substituting equations (3) and (10) into equation (1) yields

$$\rho\frac{\partial^2 V}{\partial t^2} + 2\rho c\frac{\partial^2 V}{\partial t\,\partial x} + \left(\rho c^2 - \frac{T}{A}\right)\frac{\partial^2 V}{\partial x^2} = E^*(\tfrac{1}{2}V_x^2)\,V_{xx} + V_x\{E^*(\tfrac{1}{2}V_x^2)\}_x. \tag{11}$$

Equation (11) has the same form as the equation of motion for moving elastic materials proposed by reference [4]. The difference is that the usual modulus of elasticity E is replaced by E^* which is a linear differential operator characterizing the viscoelastic property of the belt material. The differential operator E^* determined from viscoelastic models complicates the equations substantially.

Introducing the non-dimensional parameters

$$v = \frac{V}{L}, \qquad \xi = \frac{x}{L}, \qquad \tau = t\left(\frac{T}{\rho A L^2}\right)^{1/2}, \qquad \gamma = c\left(\frac{\rho A}{T}\right)^{1/2}, \qquad E = \frac{E^* A}{T},$$

the following non-dimensional equation of transverse motion can be obtained:

$$\frac{\partial^2 v}{\partial \tau^2} + 2\gamma\frac{\partial^2 v}{\partial\tau\,\partial\xi} + (\gamma^2 - 1)\frac{\partial^2 v}{\partial \xi^2} = N(v), \tag{12}$$

where the non-linear operator $N(v)$ is defined as

$$N(v) = E(\tfrac{1}{2}v_\xi^2)v_{\xi\xi} + v_\xi\{E(\tfrac{1}{2}v_\xi^2)\}_\xi. \tag{13}$$

Equations (12) and (13) are the generalized equations of motion valid for all kinds of viscoelastic model. As a first step, the most frequently used Kelvin viscoelastic model is chosen to describe the viscoelastic property of the belt material in this paper. This model is composed of a linear spring and a linear dashpot connected in parallel. The corresponding linear differential operator E^* for Kelvin viscoelastic model is

$$E^* = E_0 + \eta\frac{\partial}{\partial t}, \tag{14}$$

where E_0 is the stiffness constant of the spring and η is the dynamic viscosity of the dashpot. According to the definition of non-dimensional parameters, the dimensionless operator E can be expressed as

$$E = E_e + E_v\frac{\partial}{\partial\tau}, \tag{15}$$

where

$$E_e = E_0 A/T, \qquad E_v = \eta\sqrt{\frac{A}{\rho T L^2}}. \tag{16, 17}$$

Substituting equation (15) into equation (13), and with some manipulations, the non-linear operator $N(v)$ for the Kelvin viscoelastic model becomes

$$N(v) = \frac{3}{2}E_e v_\xi^2 v_{\xi\xi} + E_v\frac{\partial}{\partial\tau}\left(\frac{1}{2}v_\xi^2\right)v_{\xi\xi} + v_\xi E_v\frac{\partial}{\partial\tau}(v_\xi v_{\xi\xi}). \tag{18}$$

It should be mentioned that the non-linear operators in equation (18) for the Kelvin model are due to the geometric non-linearity.

Introduce the mass, gyroscopic, and linear stiffness operators as follows:

$$M = I, \qquad G = 2\gamma \frac{\partial}{\partial \xi}, \qquad K = (\gamma^2 - 1)\frac{\partial^2}{\partial \xi^2}, \tag{19}$$

where operators M and K are symmetric and positive definite for sub-critical transport speeds; G is skew-symmetric and represents a convective Coriolis acceleration component. Thus, equation (12) can be written in a standard symbolic form as

$$Mv_{\tau\tau} + Gv_\tau + Kv = N(v). \tag{20}$$

3. NON-LINEAR FREE VIBRATION ANALYSIS

In this section, non-linear vibration analysis will be performed to obtain free response and natural frequencies of viscoelastic moving belts. The method of multiple scales [12] is applied directly to the governing equations of motion without *a priori* assumption regarding the spatial solutions. Introducing a small dimensionless parameter ε as a bookkeeping device, equation (20) can be rewritten as

$$Mv_{\tau\tau} + Gv_\tau + Kv = \varepsilon N(v). \tag{21}$$

A second order uniform approximation is sought in the form

$$v(\xi, \tau, \varepsilon) = v_0(\xi, T_0, T_1) + \varepsilon v_1(\xi, T_0, T_1) + \cdots, \tag{22}$$

where $T_0 = \tau$ is a fast scale characterizing motions occurring at one of the natural frequencies ω_k of the system, and $T_1 = \varepsilon\tau$ is a slow scale characterizing the shift in the natural frequencies due to the non-linearity.

Using the chain rule, the time derivatives in terms of T_0 and T_1 become

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \cdots, \qquad \frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \cdots. \tag{23, 24}$$

Substituting equations (22)–(24) into equation (21) and equating coefficients of like powers of ε gives

$$M\frac{\partial^2 v_0}{\partial T_0^2} + G\frac{\partial v_0}{\partial T_0} + Kv_0 = 0, \qquad v_0 = 0 \quad \text{at } \xi = 0 \text{ and } 1; \tag{25, 26}$$

$$M\frac{\partial^2 v_1}{\partial T_0^2} + G\frac{\partial v_1}{\partial T_0} + Kv_1 = -2M\frac{\partial^2 v_0}{\partial T_0 \partial T_1} - G\frac{\partial v_0}{\partial T_1} + N(v_0), \tag{27}$$

$$v_1 = 0 \quad \text{at } \xi = 0 \text{ and } 1. \tag{28}$$

The excitation components on the right side of equation (27) are evaluated at the first order solution v_0 and are known at each level of approximation. The non-linear operator $N(v_0)$ in equation (27) acts on the first order correction to the displacement and velocity fields.

Equation (25) is satisfied by

$$v_0 = \psi_k(\xi)A_k(T_1)\,\mathrm{e}^{i\omega_k T_0} + \bar{\psi}_k(\xi)\bar{A}_k(T_1)\,\mathrm{e}^{-i\omega_k T_0}, \tag{29}$$

where $\psi_k(\xi)$ is the kth complex eigenfunction of the displacement field, ω_k is the kth natural frequency of the system (see Appendix), and the overbar denotes complex conjugate. Function A_k will be determined by eliminating the secular terms from v_1. Equation (29) corresponds to the free response of the unperturbed system, equation (25), in the kth mode

Substituting equation (29) into equation (27) leads to

$$M\frac{\partial^2 v_1}{\partial T_0^2}+G\frac{\partial v_1}{\partial T_0}+Kv_1=M_{1k}(E_e+2i\omega_k E_v)A_k^3\,\mathrm{e}^{3i\omega_k T_0}$$
$$+\left[M_{2k}(3E_e+2i\omega_k E_v)A_k^2\bar{A}_k-2i\omega_k A_k' M\psi_k-A_k' G\psi_k\right]\mathrm{e}^{i\omega_k T_0}+cc, \qquad (30)$$

where cc denotes the complex conjugate of all preceding terms on the right side of equation (30), the prime indicates the derivative with respect to T_1 and M_{1k}, M_{2k} are non-linear spatial operators which are defined as

$$M_{1k}=\frac{3}{2}\left(\frac{\partial\psi_k}{\partial\xi}\right)^2\frac{\partial^2\psi_k}{\partial\xi^2}, \qquad M_{2k}=\frac{1}{2}\left[\left(\frac{\partial\psi_k}{\partial\xi}\right)^2\frac{\partial^2\bar{\psi}_k}{\partial\xi^2}+2\frac{\partial\psi_k}{\partial\xi}\frac{\partial\bar{\psi}_k}{\partial\xi}\frac{\partial^2\psi_k}{\partial\xi^2}\right]. \qquad (31, 32)$$

Equation (30) has a solution only if a solvability condition is satisfied. This solvability condition demands that the right side of equation (30) be orthogonal to every solution of the homogeneous problem. For the case where internal resonance does not exist, the solvability condition can be determined as

$$-2i\omega_k A_k' m_k-A_k' g_k i+(3E_e+2i\omega_k E_v)A_k^2\bar{A}_k m_{2k}=0 \qquad (33)$$

in which

$$m_k=\langle M\psi_k,\psi_k\rangle, \qquad g_k=-i\langle G\psi_k,\psi_k\rangle,$$
$$m_{2k}=\langle M_{2k},\psi_k\rangle, \qquad (34\text{–}36)$$

and the notation $\langle\cdot,\cdot\rangle$ represents the standard inner product of two complex functions over $\xi\in(0,1)$.

Referring to Wickert and Mote [2], the kth natural frequency and eigenfunction which has been normalized such that $m_k=1$ for linear moving belts are

$$\omega_k=k\pi(1-\gamma^2), \qquad \psi_k=\sqrt{2}\sin(k\pi\xi)\,\mathrm{e}^{(ik\pi\gamma\xi)}. \qquad (37, 38)$$

The complex eigenfunctions indicate that unlike non-gyroscopic linear systems, the material particles comprising axially moving continua do not pass through equilibrium simultaneously.

Substituting the eigenvalues and eigenfunctions given by equations (37) and (38) into equations (35) and (36) leads to

$$g_k=2k\pi\gamma^2, \qquad m_{2k}=-\tfrac{1}{4}\pi^4k^4(3+2\gamma^2+3\gamma^4). \qquad (39, 40)$$

It can be seen that both g_k and m_{2k} are real.

Express A_k in the polar form

$$A_k=\tfrac{1}{2}\alpha_k\,\mathrm{e}^{i\beta_k}. \qquad (41)$$

Note that α_k and β_k represent the amplitude and the phase angle of the response, respectively.

Substituting equation (41) into equation (33) and separating the resulting equation into real and imaginary parts yield

$$\tfrac{1}{2}\alpha_k\beta_k'(2\omega_k+g_k)+\frac{\alpha_k^3}{8}(3E_e\,\mathrm{Re}\,(m_{2k})-2\omega_k E_v\,\mathrm{Im}\,(m_{2k}))=0, \qquad (42)$$

$$-\tfrac{1}{2}\alpha_k'(2\omega_k+g_k)+\frac{\alpha_k^3}{8}(3E_e\,\mathrm{Im}\,(m_{2k})+2\omega_k E_v\,\mathrm{Re}\,(m_{2k}))=0, \qquad (43)$$

where Re (m_{2k}) and Im (m_{2k}) denote the real and imaginary components of m_{2k}. Since m_{2k} is real, Im (m_{2k}) should be zero for viscoelastic moving belts.

Equation (43) is an ordinary differential equation involving one variable α_k only. After some manipulations, equation (43) can be rewritten as

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}T_1} = C_k \alpha_k^3, \tag{44}$$

where

$$C_k = \frac{3E_e \operatorname{Im}(m_{2k}) + 2\omega_k E_v \operatorname{Re}(m_{2k})}{4(2\omega_k + g_k)}. \tag{45}$$

For viscoelastic moving belts, substituting equations (39) and (40) into equation (45) leads to

$$C_k = -\frac{1}{16}\pi^4 k^4 (1-\gamma^2)(3+2\gamma^2+3\gamma^4)E_v. \tag{46}$$

Therefore, α_k can be obtained from equation (44) in the form

$$\alpha_k = \frac{\alpha_0}{\sqrt{1-2C_k\alpha_0^2 T_1}}, \tag{47}$$

where α_0 is the initial amplitude.

Substituting equation (46) into equation (47), the response amplitude of viscoelastic moving belts with geometric non-linearity can be written in the form

$$\alpha_k = \frac{\alpha_0}{\sqrt{1+\dfrac{k^4\pi^4 E_v (1-\gamma^2)(3+2\gamma^2+3\gamma^4)\alpha_0^2 \varepsilon\tau}{8}}}. \tag{48}$$

It should be noted that for the linear elastic constitutive law which does not account for damping, the amplitude α_k is a constant. However, for a viscoelastic model which takes account the damping of belt materials, the amplitude α_k should decrease with time and thus $\alpha_k' \neq 0$.

Substituting equation (47) into equation (42) gives

$$\frac{\mathrm{d}\beta_k}{\mathrm{d}T_1} = -\frac{D_k\alpha_0^2}{1-2C_k\alpha_0^2 T_1}, \tag{49}$$

where

$$D_k = \frac{3E_e \operatorname{Re}(m_{2k}) - 2\omega_k E_v \operatorname{Im}(m_{2k})}{4(2\omega_k + g_k)}. \tag{50}$$

For viscoelastic moving belts, substitution of equation (40) into equation (50) yields

$$D_k = -\tfrac{3}{32} E_e \pi^3 k^3 (3+2\gamma^2+3\gamma^4). \tag{51}$$

Solving equation (49), the solution can be expressed as

$$\beta_k = \frac{D_k}{2C_k} \ln(1-2C_k\alpha_0^2 T_1) + \beta_{k0} (C_k \neq 0), \tag{52}$$

$$\beta_k = -D_k\alpha_0^2 T_1 + \beta_{k0} (C_k = 0), \tag{53}$$

where β_{k0} is a constant.

Now that α_k, β_k and thus A_k are obtained, the first order asymptotic solution for the free vibration of moving viscoelastic belts can be obtained:

$$v_0 = \tfrac{1}{2}\psi_k(\xi)\frac{\alpha_0}{\sqrt{1-2C_k\alpha_0^2\varepsilon\tau}}\,\mathrm{e}^{i(\omega_k\tau + (D_k/2C_k)\ln(1-2C_k\alpha_0^2\varepsilon\tau)+\beta_{k0})} + cc \; (C_k \neq 0), \tag{54}$$

$$v_0 = \frac{1}{2}\psi_k(\xi)\alpha_0\,\mathrm{e}^{i((\omega_k - D_k\alpha_0^2\varepsilon)\tau + \beta_{k0})} + cc \; (C_k = 0). \tag{55}$$

Note that equation (55), for $C_k = 0$, corresponds to the linear elastic model.

Equation (54) shows clearly that the first order asymptotic solution is not a simple harmonic motion due to existence of material damping introduced by the viscoelastic model. If the material has light damping, the value of $2C_k\alpha_0^2$ is very small. In this case, the non-linear frequency ω_{Nk} can be approximated as

$$\omega_{Nk} = \omega_k - D_k\alpha_0^2\varepsilon. \tag{56}$$

Using equation (51), for light damping, the natural frequency of the viscoelastic geometric non-linear moving string is derived from equation (55) as

$$\omega_{Nk} = k\pi(1-\gamma^2) + \frac{3E_e\alpha_0^2(k\pi)^3}{32}(3 + 2\gamma^2 + 3\gamma^4). \tag{57}$$

It can be seen that the non-linear natural frequency of the system for the first order approximation is independent of the viscoelastic characteristic of the material when the Kelvin model is adopted. This is not surprising, as the frequencies of lightly damped viscoelastic materials should approach that of the elastic materials.

It follows from equation (33) that

$$A'_k = \frac{(3E_e + 2i\omega_k E_v)m_{2k}}{2i\omega_k m_k + ig_k}A_k^2\overline{A}_k. \tag{58}$$

Substituting equation (58) into equation (30), the resulting equation can be rewritten as

$$M\frac{\partial^2 v_1}{\partial T_0^2} + G\frac{\partial v_1}{\partial T_0} + Kv_1 = f_1(\xi)A_k^3\,\mathrm{e}^{3i\omega_k T_0} + f_2(\xi)A_k^2\overline{A}_k\,\mathrm{e}^{i\omega_k T_0} + cc, \tag{59}$$

where

$$f_1(\xi) = M_{1k}(E_e + 2i\omega_k E_v), \tag{60}$$

$$f_2(\xi) = M_{2k}(3E_e + 2i\omega_k E_v) - (2i\omega_k M\psi_k + G\psi_k)\frac{(3E_e + 2i\omega_k E_v)m_{2k}}{2i\omega_k m_k + ig_k}. \tag{61}$$

The solution of equation (59), which is the corresponding response correction of v_0, can be obtained using separation of variables,

$$v_1 = h_1(\xi)A_k^3\,\mathrm{e}^{3i\omega_k\tau} + h_2(\xi)A_k^2\overline{A}_k\,\mathrm{e}^{i\omega_k\tau} + cc, \tag{62}$$

where

$$h_1(\xi) = \sum_{n=\pm1,\pm2\ldots}\frac{\langle f_1(\xi),\psi_n(\xi)\rangle}{1-\dfrac{3\omega_k}{\omega_n}}\psi_n(\xi), \qquad h_2(\xi) = \sum_{\substack{n=\pm1,\pm2\ldots\\ n\neq k}}\frac{\langle f_2(\xi),\psi_n(\xi)\rangle}{1-\dfrac{\omega_k}{\omega_n}}\psi_n(\xi). \tag{63, 64}$$

The specification of no internal resonance requires that ω_k/ω_n and $3\omega_k/\omega_n$ be away from unity.

Examining equations (62)–(64), it can be seen that the spatial variations of the first order solutions are different from those of the linear solutions. Hence, the validity of the assumption that the spatial variation can be represented in terms of linear eigenfunctions is questionable. However, this assumption is the basis for the usual perturbation approach in which the partial differential equation is discretized first using linear eigenfunctions.

4. RESULTS AND DISCUSSION

Numerical results for the free vibration of viscoelastic moving belts are presented in this section. Effects of moviing speed, non-linearity and viscoelasticity are investigated. In the vibration analysis of moving belts, many people [6, 13, 14] adopt the "quasi-static stretch" assumption under which dynamic tensions in the belt are uniform throughout the span. When $T \ll EA$, the quasi-static stretch assumption is valid. In this case, the axial wave spreads much faster than the transverse wave. Thus, the variation of axial stress can be approximated to spread instantly from one end to the other. Using the method of multiple scales, free responses with this assumption are obtained to compare with those given in previous sections. Redefined M_{1k}, M_{2k}, m_{2k}, C_k, and D_k for the case with quasi-static assumption are

$$M_{1k} = \frac{\partial^2 \psi_k}{2\partial \xi^2} \int_0^1 \left(\frac{\partial \psi_k}{\partial \xi} \right)^2 \mathrm{d}\xi, \tag{65}$$

$$M_{2k} = \frac{\partial^2 \psi_k}{3\partial \xi^2} \int_0^1 \left(\frac{\partial \psi_k}{\partial \xi} \right)\left(\frac{\partial \overline{\psi}_k}{\partial \xi} \right) \mathrm{d}\xi + \frac{\partial^2 \overline{\psi}_k}{6\partial \xi^2} \int_0^1 \left(\frac{\partial \psi_k}{\partial \xi} \right)^2 \mathrm{d}\xi, \tag{66}$$

$$m_{2k} = -\frac{k^2\pi^2}{6\xi^2}(2k^2\pi^2\gamma^2(\gamma^2+1)^2 + \sin^2(k\pi\gamma)), \tag{67}$$

$$C_k = -\frac{E_v \pi^2 k^2 (1-\gamma^2)(2\pi^2 k^2 \gamma^2 (1+\gamma^2)^2 + \sin^2(k\pi\gamma))}{24\gamma^2}, \tag{68}$$

$$D_k = -\frac{E_e \pi k (2\pi^2 k^2 \gamma^2 (1+\gamma^2)^2 + \sin^2(k\pi\gamma))}{16\gamma^2}. \tag{69}$$

Applying equations (54) and (62), the natural frequencies and response amplitudes of viscoelastic moving belts with geometric non-linearity under the quasi-static assumption are derived as

$$\omega_{Nk} = k\pi(1-\gamma^2) + \frac{E_e \alpha_0^2 (k\pi)^3}{8}\left[(1+\gamma^2)^2 + \frac{1}{2}\left(\frac{\sin(k\pi\gamma)}{k\pi\gamma}\right)^2\right], \tag{70}$$

$$\alpha_k = -\frac{\alpha_0}{\sqrt{1 + [E_v \pi^2 k^2 \alpha_0^2 (1-\gamma^2)(2\pi^2 k^2 \gamma^2 (1+\gamma^2)^2 + \sin^2(k\pi\gamma))\varepsilon\tau]/12\gamma^2}}. \tag{71}$$

It can be seen that for viscoelastic materials, the natural frequencies given by equation (70) are identical to those of elastic systems [6]. However, the response amplitudes depend on the viscoelastic property of the material. When the linear elastic model is considered, in which $E_v = 0$, the response amplitude remains constant. This conclusion agrees with reference [6].

Figure 4 compares the fundamental natural frequencies without the quasi-static assumption and those with the quasi-static assumption for moving belts. The non-linear natural frequency is plotted against the non-dimensional transport speed. The material is assumed to be linear elastic, i.e., $E_v = 0$. Different values of E_e are chosen to show the influence of non-linearity. The higher the value of E_e, the stronger the non-linearity of the system. It is observed that the natural frequency decreases as the transport speed increases. This is because a larger moving speed leads to a smaller linear stiffness of the belt, resulting in lower frequencies. Note that with the increase in the non-linearity, the natural frequency increases. It can be seen that the results with the quasi-static assumption and those without such an assumption are close to each other at the lower speed range. The difference, however, grows with the moving speed. This is because at a higher moving speed, the contribution of the non-linearity to the natural frequencies is larger. Since the quasi-static assumption only involves the non-linear terms, the difference between non-linear terms with the quasi-static assumption and those without quasi-static assumption leads to a larger difference of natural frequencies.

To show the influence of viscoelastic parameter E_v on non-linear natural frequencies, equation (54) is rewritten in the near- and exact-resonance conditions as

$$v_0 = \zeta_k^R \psi_k^R + \zeta_k^I \psi_k^I, \tag{72}$$

where ζ_k^R and ζ_k^I are generalized co-ordinates governing the evolution of the real and imaginary components of the eigenfunction ψ_k, ζ_k^R and ζ_k^I, and can be derived as

$$\zeta_k^R = \frac{\alpha_0}{\sqrt{1 - 2C_k \alpha_0^2 \varepsilon\tau}} \cos\left(\omega_k \tau + \frac{D_k}{2C_k} \ln(1 - C_k \alpha_0^2 \varepsilon\tau) + \beta_{k0}\right), \tag{73}$$

$$\zeta_k^I = \frac{\alpha_0}{\sqrt{1 - 2C_k \alpha_0^2 \varepsilon\tau}} \sin\left(\omega_k \tau + \frac{D_k}{2C_k} \ln(1 - C_k \alpha_0^2 \varepsilon\tau) + \beta_{k0}\right). \tag{74}$$

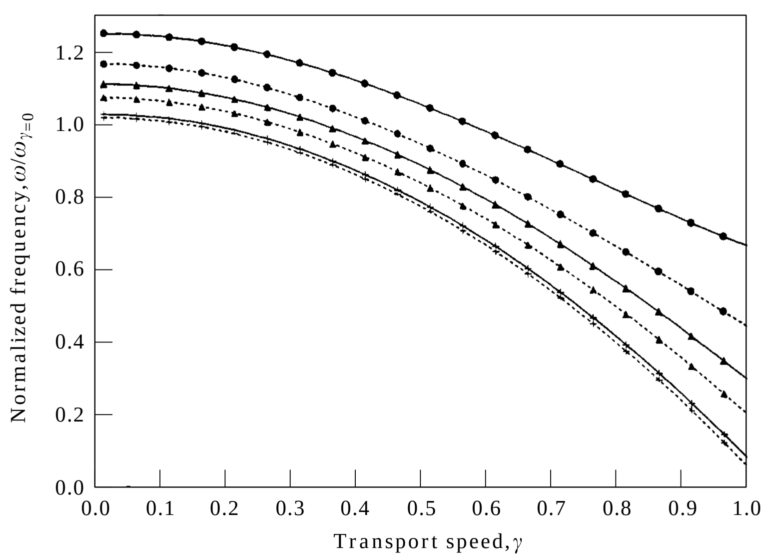

Figure 4. A comparison of non-linear fundamental frequencies of an elastic moving belt ($\alpha_0 = 0{\cdot}005$): —, without quasi-static assumption; · · · · · , with quasi-static assumption; +, $E_e = 400$; ▲, $E_e = 1600$; ●, $E_e = 3600$.

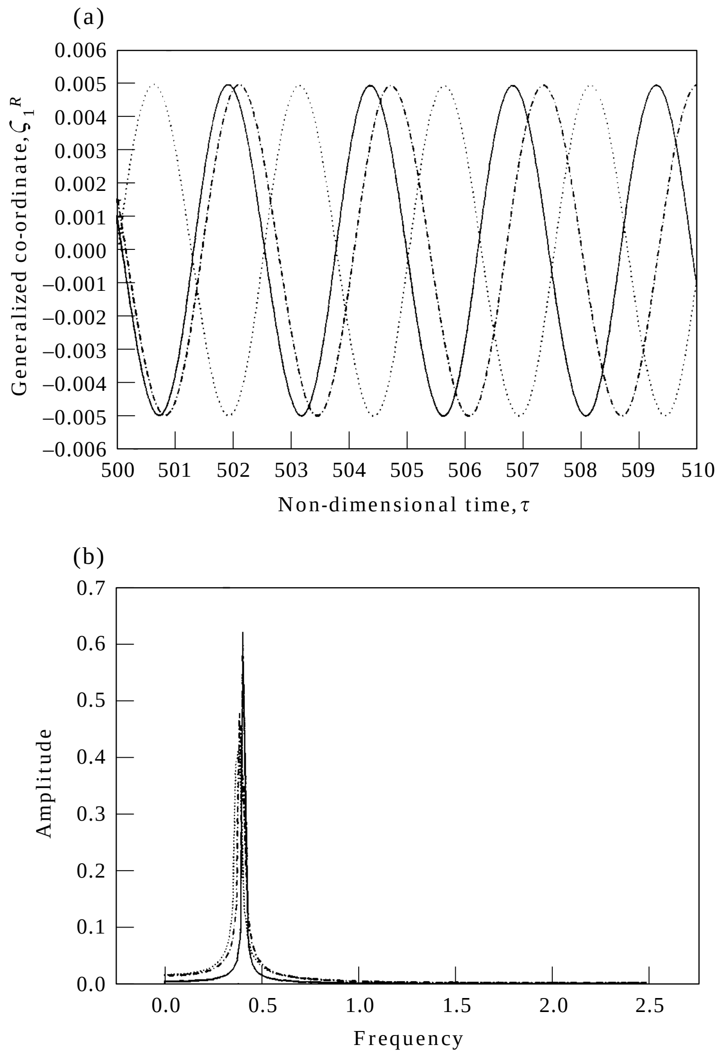

Figure 5. (a) Waveform of generalized co-ordinate ζ_l^R and (b) the discrete Fourier transform of the waves for $E_e = 400$, $\gamma = 0{\cdot}5$: —, $E_v = 0{\cdot}1$; – · –, $E_v = 1$; · · · · · ·, $E_v = 10$.

ζ_k^R and ζ_k^I in the time domain and the frequency domain for different values of E_v are displayed in Figures 5 and 6, respectively, to show effects of viscoelasticity on natural frequencies. The viscoelastic parameters E_v are chosen as 0·1, 1 and 10, respectively. Since ζ_k^R and ζ_k^I are not of simple harmonic motion, it is difficult to show the influence of E_v on natural frequencies during a short time scale. Thus, in Figures 5 and 6, the starting point of the non-dimensional time is chosen as 500. It can be seen that even after a longer time, the difference of natural frequencies among different values of E_v is still small. Therefore, for a first order approximation, the viscoelasticity does not have a significant effect on the natural frequency of viscoelastic moving belts. Materials with strong damping that do not suffer from greatly reduced natural frequencies are conceivable.

The effect of the viscoelastic parameter E_v on the response amplitude α_k is illustrated in Figures 7 and 8. In Figure 7, the response amplitude is plotted over the non-dimensional time range 0–500. E_e and the non-dimensional transport speed γ are set to be 400 and 0·5, respectively. Three different values of viscoelastic parameter E_v are considered. At the time instant $\tau = 500$, the amplitude decrease is 2% for $E_v = 0{\cdot}1$, 16·2% for $E_v = 1$ and 56·2% for $E_v = 10$. As expected, the response amplitude decreases with time and the amplitude is strongly dependent on the viscoelastic coefficient E_v. In Figure 8, the response amplitude is plotted over the viscoelastic parameter E_v ranging from 0 to 50 at the time instant $\tau = 500$ while other parameters remain the same as those in Figure 7. It can be seen that the larger E_v is, the smaller the amplitude. Since higher E_v correspond to higher damping, the viscoelastic nature of the material can be effective in reducing the vibration of moving belts.

Figure 9 shows the effect of non-linearity, reflected by E_e, on response amplitudes for viscoelastic moving belts. Three viscoelastic systems having identical E_v and γ but having different E_e are compared: $e_v = 1$ and $\gamma = 0{\cdot}5$. For system 1, $E_e = 400$; for system 2, $E_e = 2500$; for system 3, $E_e = 10\,000$. It is clear that the amplitudes are identical for three different systems over the non-dimensional time range 0–500. Hence, it is concluded that

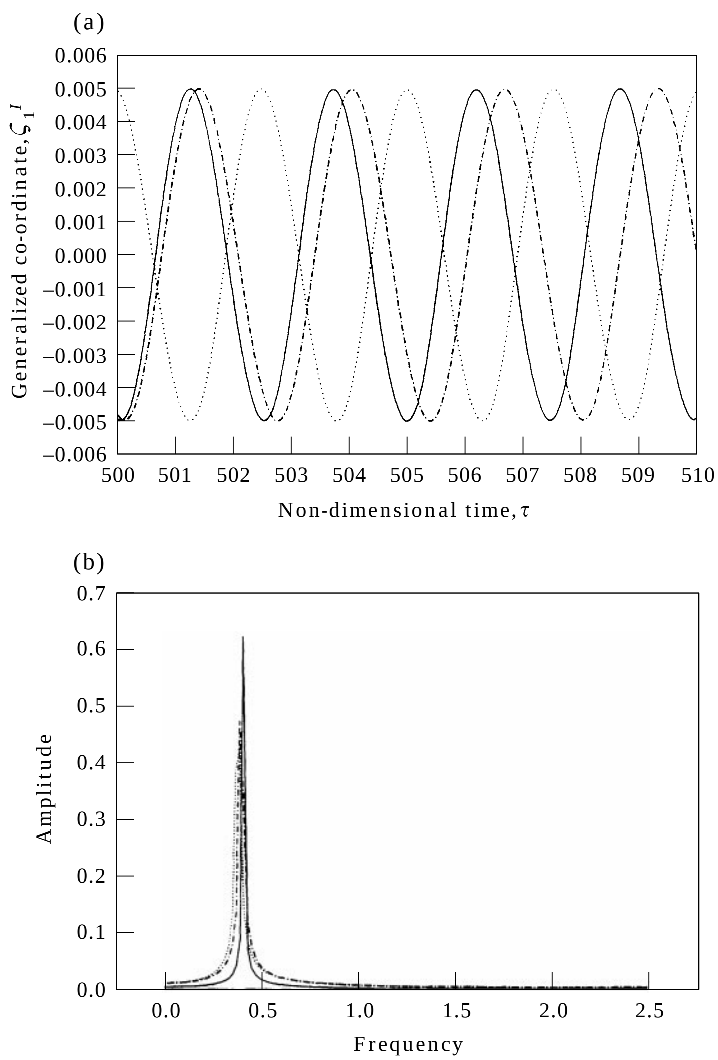

Figure 6. (a) Waveform of generalized co-ordinate ζ_1^I and (b) the discrete Fourier transform of the waves for $E_e = 400$, $\gamma = 0{\cdot}5$: —, $E_v = 0{\cdot}1$; – · –, $E_v = 1$; . . . . . . , $E_v = 10$.

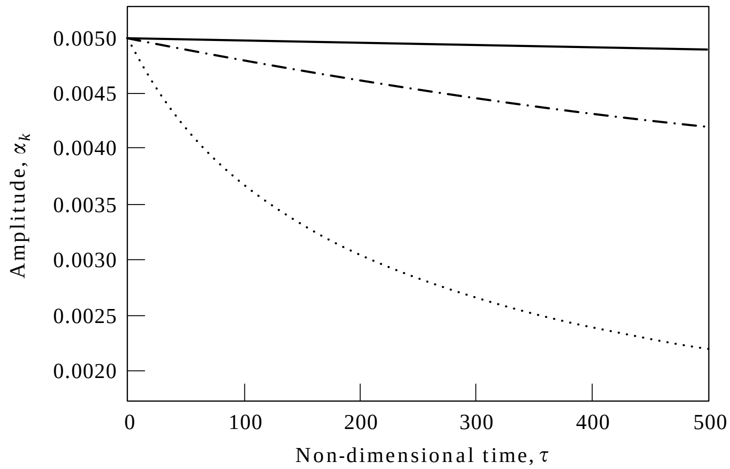

Figure 7. The influence of viscoelasticity on response amplitudes for $E_e = 400$, $\gamma = 0{\cdot}5$: —, $E_v = 0{\cdot}1$; – · – · –, $E_v = 1$; · · · · · ·, $E_v = 10$.

the non-linear parameter E_e has no influence on the amplitude of response while E_e affects the non-linear natural frequencies of viscoelastic moving belts, as shown in Figure 4.

5. CONCLUSIONS

In this paper, the non-linear natural frequencies and near-modal non-linear response for free vibration of viscoelastic elastic moving belts are obtained by using the method of multiple scales. The Kelvin model is adopted to describe the viscoelastic characteristic of belt materials. The governing equaion of motion is derived and cast in a first order form. The method of multiple scales is applied directly to the governing partial differential equation. The effects of axial moving speed, geometric non-linearity and viscoelastic property on the natural frequencies and amplitudes of free response are investigated from the numerical examples. The following conclusions are made in this study:

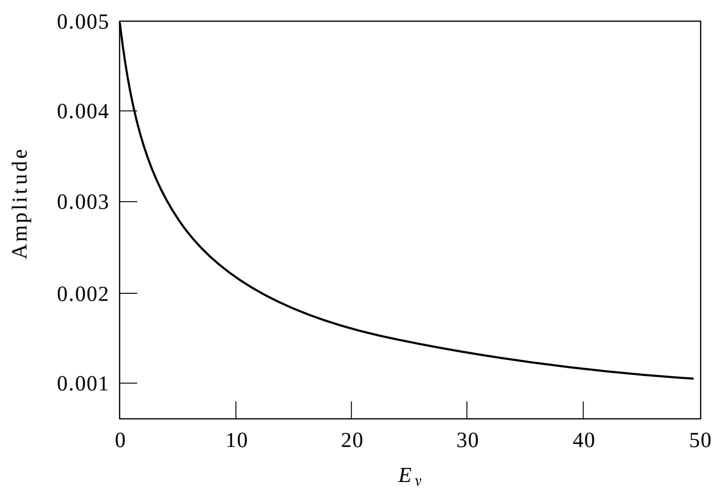

Figure 8. The influence of viscoelasticity on response amplitudes for $E_e = 400$, $\gamma = 0{\cdot}5$ at time instant 500.

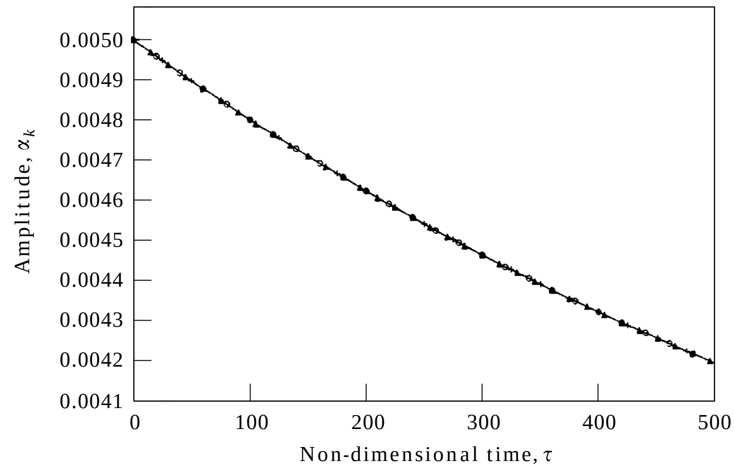

Figure 9. The influence of non-linearity on response amplitudes for $E_v = 1$, $\gamma = 0{\cdot}5$: $\times$, $E_v = 400$; ▲, $E_e = 1600$; ○, $E_e = 3600$.

(1) The damping introduced by the viscoelastic model has no significant effect on non-linear natural frequencies while it has an important influence on the amplitude of response for viscoelastic moving belts. The response amplitude decreases with time more quickly than the increase of the viscoelastic coefficient E_v. Thus, materials with strong viscoelastic property can effectively reduce the vibration of moving belts without suffering from greatly reduced natural frequencies.

(2) The non-linear natural frequencies decrease as the moving speed increases.

(3) The natural frequencies grow with the non-linear parameter E_e, but the free response amplitude does not change with E_e.

(4) No assumptions about the spatial dependence of the motion are made in the method of solution. This is more appropriate than usual perturbation approaches in which the linear spatial solutions are assumed *a priori* to describe the spatial solution of the non-linear problem.

(5) The method of solution can be applied to a wide range of gyroscopic systems and general linear viscoelastic materials without being restricted to moving materials and the Kelvin model.

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REFERENCES

1. R. Skutch 1897 *Ann. Phys. Chem.* **61**, 190–195. Üeber die Bewegung eines gespannten Fadens.
2. J. A. Wickert and C. D. Mote Jr 1990 *ASME Journal of Applied Mechanics* **57**, 738–744. Classical vibration analysis of axially moving continua.
3. C. D. Mote Jr 1966 *ASME Journal of Applied Mechanics* **33**, 463–464. On the non-linear oscillation of an axially moving string.
4. A. L. Thurman and C. D. Mote Jr 1969 *ASME Journal of Applied Mechanics* **36**, 83–91. Free, periodic, non-linear oscillation of an axially moving strip.
5. V. A. Bapat and P. Srinivasan 1967 *ASME Journal of Applied Mechanics* **34**, 775–777. Non-linear transverse oscillation in traveling strings by the method of harmonic balance.

6. J. A. Wickert 1992 *International Journal of Non-linear Mechanics* **27**, 503–517. Non-linear vibration of a traveling tensioned beam.
7. H. Palmgren 1986 *The V-belt Handbook*. Sweden Studentliteratur.
8. W. N. Findley, L. S. Lai and K. Onarna 1976 *Creep and Relaxation of Nonlinear Viscoelastic Materials*. New York: North-Holland. See p. 109.
9. R. M. Christensen 1982 *Theory of Viscoelasticity*. New York: Academic Press; second edition. See p. 24.
10. T. M. Chen 1995 *International Journal for Numerical Method in Engineering* **38**, 509–522. The hybrid Laplace transform/finite element method applied to the quasi-static and dynamic analysis of viscoelastic Timoshenko beams.
11. R. F. Fung, J. S. Huang and Y. C. Chen 1997 *Journal of Sound and Vibration* **201**, 153–167. The transient amplitude of the viscoelastic traveling string: an integral constitutive law.
12. A. H. Nayfeh and S. A. Nayfeh 1994 *Journal of Vibration and Acoustics* **116**, 129–136. On nonlinear modes of continuous systems.
13. J. S. Huang, R. F. Fung and C. H. Lin 1995 *International Journal of Mechanical Sciences* **37**, 145–160. Dynamic stability of a moving string undergoing three-dimensional vibration.
14. E. M. Mockensturm, N. C. Perkins and A. G. Ulsoy 1994 *Nonlinear and Stochastic Dynamics* **78**, 31–36. Stability and limit cycles of parametrically excited, axially moving string.

APPENDIX: CLOSED-FORM SOLUTION

The closed-form solution for the vibration analysis of linear moving materials is presented [2] in this appendix. The base solution is used in the non-linear vibration analysis of moving belts.

Introduce the state vector and the excitation vectors

$$\mathbf{w} = \begin{Bmatrix} v_\tau \\ v \end{Bmatrix}, \qquad \mathbf{q} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}, \tag{A1}$$

and the matrix differential operators

$$\mathbf{A} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} G & K \\ -K & 0 \end{bmatrix}. \tag{A2}$$

Equation (20) without the non-linear term becomes

$$\mathbf{A}\mathbf{w}_\tau + \mathbf{B}\mathbf{w} = \mathbf{q}. \tag{A3}$$

Equation (A3) is a canonical form of the equation of motion and its solution satisfies the initial condition v_0 and the corresponding boundary condition. The general solution of the linear response for equation (A3) is

$$\mathbf{w}(\xi, \tau) = \sum_{n = \pm 1, \pm 2, \ldots} \zeta_n(\tau)\boldsymbol{\phi}_n(\xi), \tag{A4}$$

where

$$\zeta_n(\tau) = \zeta_n(0)\,\mathrm{e}^{\lambda_n \tau} + \int_0^\tau \mathrm{e}^{\lambda_n(\tau - t)} q_n(t)\,\mathrm{d}t, \qquad q_n(\tau) = \langle \mathbf{q}, \boldsymbol{\phi}_n \rangle, \qquad \zeta_m(0) = \langle \mathbf{A}\mathbf{w}_0, \boldsymbol{\phi}_n \rangle \tag{A5–A7}$$

and the eigenvalues $\lambda_n = i\omega_n$ are imaginary with natural frequencies ω_n being positive for $n \geqslant 1$; $\boldsymbol{\phi}_n(\xi)$ is the state eigenfunction that has the representation $\boldsymbol{\phi}_n = \{\lambda_n\psi_n, \psi_n\}^T$ in terms of the complex scalar eigenfunction ψ_n of the displacement field; $\boldsymbol{\phi}_n(\xi)$ satisfies the orthogonal relations

$$\langle \mathbf{A}\boldsymbol{\phi}_n, \boldsymbol{\phi}_m \rangle = \delta_{mn}, \qquad \langle \mathbf{B}\boldsymbol{\phi}_n, \boldsymbol{\phi}_m \rangle = -\lambda_n \delta_{mn}, \qquad \text{for } n, m = \pm 1, \pm 2, \ldots . \tag{A8}$$

In particular, the closed form steady-state displacement response for the non-resonance harmonic excitation $\mathbf{q} = \{f(x)\,\mathrm{e}^{i\omega\tau} \;\; 0\}^T$ is

$$v(\xi, \tau) = \mathrm{e}^{i\omega\tau} \sum_{n=\pm 1, \pm 2, \ldots} \frac{\langle f, \psi_n \rangle}{1 - \omega/\omega_n} \psi_n(\xi). \tag{A9}$$